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Schauder Bases in Banach Spaces: Application to Numerical Solutions of Differential Equations

A. PALOMARES, M. PASADAS, V. RAMÍREZ AND M. RUIZ GALÁN
Dpto. Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain
<anpalom><mpasadas><vramirez><mruizg>@ugr.es

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Abstract—In this work, we present a numerical method for approximating the solution of a linear differential problem. We formulate a general problem in a Banach space making use of a Schauder basis and the Tau method to approximate the load function and the solution of the differential problem. Finally, we offer a numerical example.

Keywords—Differential equations, Approximation, Schauder bases, Tau method.

1. INTRODUCTION

The solution of many problems consists in finding the inverse of a given function by means of an operator. This function is usually known as the load function in differential problems.

In this paper, we construct approximations of the load function starting from some functions for which we can easily solve the differential problem. Such functions are the elements of certain Schauder basis. In this way, the solution of the differential problem is obtained as the norm-limit of a sequence of the approximations, and an estimation of the error is established. In practice, the numerical solution usually requires calculating some particular solutions using the Tau method [1,2].

We will consider the Banach spaces $C[0,1]$ and $C^2[0,1]$ endowed with their usual norms

$$\|y\|_\infty = \max_{t\in[0,1]} |y(t)| \qquad (y \in C[0,1]),$$

and

$$\|x\|_{C^2} = \|x''\|_\infty + \|x'\|_\infty + \|x\|_\infty \qquad \left(x \in C^2[0,1]\right).$$

2. FORMULATING THE PROBLEM

Let X and Y be Banach spaces (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), let $D : X \to Y$ be a one-to-one bounded and linear operator and let $y_0 \in Y$. We consider the following problem:

$$\text{find } x_0 \in X \text{ such that } Dx_0 = y_0. \tag{P}$$

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When D is a linear differential operator, this problem constitutes a linear differential equation. Thus, a method for solving this problem is, in particular, a method for solving differential equations.

3. SCHAUDER BASIS

As mentioned in the introduction, we will make use of the concept of Schauder basis. Let us recall that a sequence $\{y_n\}_{n\geq 1}$ in a Banach space Y is a Schauder basis provided that for all $y \in Y$ there exists a unique sequence $\{a_n\}_{n\geq 1} \subset \mathbb{K}$ such that

$$y = \sum_{n=1}^{\infty} a_n y_n.$$

The scalars $a_n \in \mathbb{K}$ are called the coefficients of y in the basis $\{y_n\}_{n\geq 1}$. If, for each $n \geq 1$, $y_n^*(y)$ is the unique a_n such that $y = \sum_{n=1}^{\infty} a_n y_n$, then y_n^* is a bounded and linear functional on Y. They are called the functionals associated with the basis $\{y_n\}_{n\geq 1}$ and the sequence of bounded linear operators $P_n : Y \to Y$, given by

$$P_n y = \sum_{i=1}^{n} y_i^*(y) y_i \qquad (y \in Y)$$

are known as the sequence of projections of $\{y_n\}_{n\geq 1}$.

If $\{y_n\}_{n\geq 1}$ is an orthogonal base in a separable Banach space, then it plainly is a Schauder basis. Schauder bases are explicitly known for Banach spaces of sequences as ℓ_p $(1 \leq p < \infty)$ or c_0, and for Banach spaces of functions as $L_p([a,b])$ $(1 \leq p < \infty)$, $C^k([0,1]^d)$ or $W_p^m([0,1]^d)$ (see [3–5]).

Given a partition $T_n = \{t_0 = 0 < t_1 < \cdots < t_n = 1\}$ of the interval $[0,1]$, let us denote by $S_1(T_n)$ the $(n+1)$-dimensional linear space of all continuous spline functions of degree 1 with knots T_n and, by

$$\left\{s_{t_i}^{T_n} : i = 0, 1, \ldots, n\right\},$$

the Lagrangian basis, i.e., $s_{t_i}^{T_n}$ is the unique function in $S_1(T_n)$ such that

$$s_{t_i}^{T_n}(t_j) = \delta_{ij}, \qquad j = 0, 1, \ldots, n.$$

We can set up the usual Schauder basis in $C[0,1]$ as follows.

DEFINITION 3.1. *Let $T = \{t_n\}_{n\geq 0}$ denote a dense sequence in $[0,1]$, with $t_0 = 0$, $t_n = 1$, and $t_i \neq t_j$ for $i \neq j$. For all $n \geq 1$, let $T_n = \{t_j \mid j = 0, \ldots, n\}$. The Schauder system designates the functions $\{\varphi_n^T\}_{n\geq 0}$ given by $\varphi_0^T = s_{t_0}^{T_1}$, $\varphi_n^T = s_{t_n}^{T_n}$, for $n \geq 1$.*

For a proof of the following result, you can see [3].

PROPOSITION 3.1. *The Schauder system $\{\varphi_n\}_{n\geq 0}$ is a Schauder basis in $C[0,1]$.*

The following easy property on Schauder bases provides the solution of problem (P) as the limit of a sequence, and constitutes a numerical method for solving some ordinary differential equations.

PROPOSITION 3.2. *Let X and Y be Banach spaces, let $y_0 \in Y$ and let $D : X \to Y$ a one-to-one bounded linear operator. We assume that $\{y_n\}_{n\geq 1} \subset Y$ is a Schauder basis and $\{P_n\}_{n\geq 1}$ is the sequence of the associate projections. Then the unique solution of problem (P) is given by*

$$x_0 = \lim_{n\to\infty} D^{-1}(P_n y_0).$$

Moreover, this solution satisfies

$$\left\|x_0 - D^{-1}\left(P_n y_0\right)\right\| \leq \left\|D^{-1}\right\| \left\|y_0 - P_n y_0\right\|.$$

PROOF. By virtue of the definition of Schauder basis and the projections P_n, given $y \in Y$, we obtain that

$$\lim_{n\to\infty} \|y - P_n y\| = 0. \tag{1}$$

Let x_0 be the solution of problem (P). Then, for all $n \in \mathbb{N}$, the boundedness of the operator D^{-1} (by the inverse mapping theorem) gives that

$$\left\|x_0 - D^{-1}\left(P_n x_0\right)\right\| = \left\|D^{-1} y_0 - D^{-1}\left(P_n y_0\right)\right\| \leq \left\|D^{-1}\right\| \left\|y_0 - P_n y_0\right\|. \tag{2}$$

Finally, it follows from (1) and (2) that

$$\lim_{n\to\infty} \left\|x_0 - D^{-1}\left(P_n y_0\right)\right\| = 0.$$

4. THE APPLICATIONS

The above result allows us to calculate some numerical solutions of differential equations.

First of all, we should fix the Banach spaces X and Y and the operator $D : X \to Y$ so that the differential equation be unisolvent, i.e., D is a one-to-one and linear operator. Then the continuity of the operator D (or equivalently D^{-1}) must be proven. The boundary conditions will obviously influence the choice of X.

Afterwards, we will consider a Schauder basis $\{y_n\}_{n\geq 1}$ in the Banach space Y. Since the sequence $\{D^{-1}y_n\}_{n\geq 1}$ is a basis in X and the coefficients of y_0 with respect to $\{y_n\}_{n\geq 1}$ are the same as the ones of $x_0 = D^{-1}y_0$ in $\{D^{-1}y_n\}_{n\geq 1}$, it suffices to obtain, for all $n \in \mathbb{N}$, $D^{-1}y_n$. Such a function can then be calculated by means of the Tau method (see [1,2]).

EXAMPLE 4.1. We now apply the preceding method. We shall obtain a numerical solution of the following problem: given $y \in C[0,1]$, find $x \in C^2[0,1]$ such that

$$\begin{aligned} x''(t) + x(t) &= y(t), \qquad t \in (0,1), \\ x(0) = x(1) &= 0. \end{aligned}$$

In this case, we take as X the Banach space

$$X = \left\{x \in C^2[0,1],\ x(0) = x(1) = 0\right\},$$

with $\|\cdot\|_X = \|\cdot\|_{C^2}$; $Y = C[0,1]$, with its usual supnorm; $D : X \to Y$ the operator

$$Dx = x'' + x;$$

and $y_0 \in Y$ as the load function.

This problem has a unique solution $x_0 \in X$ for a given $y_0 \in Y$. Moreover, we have that $\|Dx\| = \|x'' + x\| \leq \|x''\|_\infty + \|x\|_\infty \leq \|x\|_X$. Thus, D is a continuous operator and, according to the inverse mapping theorem, D^{-1} it is also continuous.

We proceed to consider the Schauder system $\{\varphi_n^T\}_{n\geq 0}$. Given $n \in \mathbb{N}$,

$$P_n y_0 = \sum_{i=0}^{n} \left(\varphi_i^T\right)^* (y_0)\varphi_i^T,$$

and since

$$S_1(T_n) = \left\langle \varphi_0^T, \dots, \varphi_n^T \right\rangle,$$

then

$$P_n y_0 = \sum_{i=0}^{n} y_0(t_i) s_{t_i}^{T_n}.$$

Therefore, from Proposition 3.2, we obtain the following approximation of x_0:

$$D^{-1}(P_n y_0) = D^{-1}\left(\sum_{i=0}^{n} y_0(t_i) s_{t_i}^{T_n}\right) = \sum_{i=0}^{n} y_0(t_i) D^{-1}\left(s_{t_i}^{T_n}\right).$$

To explicitly obtain a numerical solution, we calculate $D^{-1}(s_{t_i}^{T_n})$ for $i = 0, \dots, n$, that is, we solve the differential equation for those load functions which are polygonal functions.

In the following table, we show some estimations of the approximation error $\|y_0 - P_n y_0\|_\infty$ in the first column and the approximation error of the solution $\|x_0 - D^{-1}(P_n y_0)\|_{C^2}$ in the second one, for some values of n, with $y_0(t) = 2e^t + (6-e)t + t^3 - 1$.

Table 1.

n	1	2
5	0.0516326	0.0702156
10	0.0135896	0.0180978
20	0.00348094	0.00458235
40	0.000882334	0.00115626
80	0.00021919	0.000287706

5. CONCLUSIONS

We have presented a numerical method that allows some differential equations to be solved with a low computational cost. The use of the Schauder basis provided for the convergence to the exact solution, whereas the introduction of a basis of the Lagrangian splines simplifies the calculation of the approximations.

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